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Path integral with ghosts for the bosonic string propagator

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Abstract. We compute the propagator for the open bosonic string using the Polyakov path integral formalism, both with and without ghosts.

1. Introduction

String field theory [1-7] and the Polyakov path integral [8-14] are two complementary techniques for treating problems in string physics. The former is potentially of use for studying non-perturbative questions, while the latter has the advantage of manifest duality in its basic formulation. It is thus of interest to understand, in as detailed a manner as possible, the connections between the two formalisms.

The most thoroughly studied gauge-covariant string field theory is that of Witten [2]. The free theory is a theory of open strings‖, in which the fields depend not only on the string's spacetime coordinates, but also on ghost coordinates related to the string's embedding in two-dimensional parameter space [6]. The free propagator for the Witten theory has been computed in [7].

However, to the best of our knowledge, neither the free open-string propagator, with or without ghosts, nor *any* amplitude *with* ghosts¶, has been computed using the Polyakov path integral formalism. Thus, the present work.

2. The propagator without ghosts

We want to compute to lowest order the amplitude $A(X_i \to X_f)$ for an open string to propagate from $X_i^\mu(\sigma)$ to $X_f^\mu(\sigma)$, where $X_i^\mu(\sigma)$ and $X_f^\mu(\sigma)$ are two arcs in spacetime. The Polyakov path integral method allows one to obtain the Euclidean theory amplitude $A_{\mathrm{E}}(X_i \to X_f)$ through the formula

$$A_E(X_i \to X_f) = \int \mathrm{d}\Sigma_i \,\mathrm{d}\Sigma_f \int \frac{1}{V_{\mathrm{W-GC}}} [\mathrm{D}g][\mathrm{D}X] \exp(-S) \qquad (2.1a)$$

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‖ It has been proposed that the interacting theory automatically contains closed strings as well [5].

¶ After the completion of this work we became aware of [15], where the bosonic open- and closed-string propagators are computed in a manner different from that of the present paper, and of [16], where vacuum wavefunctionals with ghosts are computed. Subsequent work on string propagators with ghosts has appeared in [17]. A brief description of the present work appeared in [24].

where S is the Polyakov action:

$$S = S[g_{ab}, X^\mu] = \tfrac{1}{2}T\int_M d^2\sigma\sqrt{g}\,g^{ab}\partial_a X^\mu \partial_b X^\mu. \tag{2.1b}$$

Here σ^1 and σ^2 are the coordinates of the string worldsheet. For definiteness, we shall consider σ^1 as a spacelike coordinate and σ^2 as a timelike coordinate, even though in Euclidean theory this distinction is purely conventional. In the case we are considering, the worldsheet M has the topology of a square. Thus, M can be parametrised in an obvious fashion by making σ^1 and σ^2 vary in the closed interval $[0, 1]$. g_{ab} is the metric of the worldsheet and g is the determinant of g_{ab}; X^μ, $\mu = 1, \ldots, 26$, are the coordinates describing the string's embedding in 26-dimensional Euclidean space R^{26}. T, the string tension, will be set equal to 1 (corresponding to $\alpha' = \frac{1}{2}\pi$ in the other common notation).

The path integral is first performed over all embeddings X^μ satisfying the Neumann boundary condition $N^a\partial_a X^\mu = 0$ at $\sigma^1 = 0, 1$ (N^a is the inward unit normal to the boundary of the worldsheet when the metric is g_{ab}), and $X^\mu = X_i^\mu \circ \Sigma_i$ at $\sigma^2 = 0$, $X^\mu = X_f^\mu \circ \Sigma_f$ at $\sigma^2 = 1$ for some reparametrisations Σ_i and Σ_f of the interval $[0, 1]$. Then, the path integral is performed over the metrics g_{ab} and the reparametrisations Σ_i and Σ_f. Physically, the Neumann boundary condition at $\sigma^1 = 0, 1$ implies that there is no net flux of momentum through the ends of the propagating string. Mathematically, we need to have a well defined classical problem. The reparametrisations Σ_i and Σ_f are introduced because, in principle, the parametrisation of the embedding X^μ at $\sigma^2 = 0, 1$ may not match that of the boundary arcs X_i^μ and X_f^μ. Since the amplitude $A_E(X_i \to X_f)$ is geometrical in nature it cannot depend on the parametrisations of X_i^μ and X_f^μ. Thus the integration over Σ_i and Σ_f must be performed.

As is well known, the Polyakov action enjoys two types of gauge symmetry: general coordinate invariance and Weyl rescaling invariance. However, the path integral measures $[Dg]$, $[DX]$ enjoy only general coordinate invariance [8-10]. We expect the resulting Weyl anomaly to cancel in 26 dimensions, as it does at the level of the partition function. While this is almost certainly true, the proof of this fact is quite difficult. Indeed, the worldsheet under consideration has corners, and these may contribute to the Weyl anomaly. Alvarez's calculation [9], therefore, does not apply. For the time being, we assume that the Weyl anomaly cancels, and divide the path integral by the infinite volume $V_{\text{W-GC}}$ of the string gauge group to get a finite result.

To compute the functional integral (2.1*a*) we must fix the gauge. The gauge-fixing procedure has been analysed extensively in the literature [8-13]. Thus we shall restrict ourselves to stating the result:

$$A_E(X_i \to X_f) = \int d\Sigma_i\, d\Sigma_f \int_0^\infty d\lambda\, \exp(-S[\hat{g}_{ab}(\lambda), X_{\text{cl}}^\mu(\lambda, \Sigma_i, \Sigma_f, X_i^\mu, X_f^\mu)]) \times (\det{}' P^\dagger P)^{1/2} \frac{(\psi_{ab}\,|\,d\hat{g}_{ab}(\lambda)/d\lambda)}{(\psi_{ab}\,|\,\psi_{ab})^{1/2}} (\det\Delta)^{-13}. \tag{2.2}$$

Here $\hat{g}_{ab}(\lambda)$ is the Teichmuller metric. For the simple topology we are considering, there is only one Teichmuller parameter λ, varying in the semi-infinite interval $]0, +\infty[$. Δ is the scalar Laplacian, defined by

$$\Delta\phi = \frac{-1}{\sqrt{\hat{g}}}\partial_a(\sqrt{(\hat{g})}\hat{g}^{ab}\partial_b\phi) \tag{2.3}$$

where ϕ is any worldsheet scalar function. The operator P maps contravariant vector fields η^a into traceless symmetric covariant 2-tensors (the trace is taken with respect to the metric $\hat{g}_{ab}(\lambda)$):

$$(P\eta)_{ab} = \hat{g}_{ac}\nabla_b\eta^c + \hat{g}_{bc}\nabla_a\eta^c - \hat{g}_{ab}\nabla_c\eta^c \tag{2.4a}$$

where ∇_a is the covariant derivative associated with the metric $\hat{g}_{ab}(\lambda)$. The operator $P^\dagger$ is the formal adjoint of P and maps traceless symmetric covariant 2-tensors ζ_{ab} into contravariant vector fields:

$$(P^\dagger\zeta)^a = -2\hat{g}^{ab}\hat{g}^{cd}\nabla_c\zeta_{db}. \tag{2.4b}$$

In the case of a square worldsheet the operator P has no zero modes (zero modes of P are called conformal Killing vectors). Conversely, the operator $P^\dagger$ has a traceless symmetric zero mode ψ_{ab} (traceless symmetric zero modes of $P^\dagger$ are called Teichmuller deformations). The symbols involving ψ_{ab} are defined as follows:

$$(\xi^{(1)}_{ab}|\xi^{(2)}_{ab}) = \int_M \mathrm{d}^2\sigma\sqrt{(\hat{g})}\hat{g}^{ab}\hat{g}^{cd}\xi^{(1)}_{ac}\xi^{(2)}_{bd} \tag{2.5}$$

where $\xi^{(i)}_{ab}$ is a covariant 2-tensor. Finally, $X^\mu_{\mathrm{cl}}(\lambda, \Sigma_i, \Sigma_f, X^\mu_i, X^\mu_f)$ is the solution of the classical problem:

$$\Delta X^\mu_{\mathrm{cl}} = 0 \tag{2.6a}$$

$$N^a\partial_a X^\mu_{\mathrm{cl}} = 0 \qquad \text{at } \sigma^1 = 0, 1 \tag{2.6b}$$

$$X^\mu_{\mathrm{cl}} = X^\mu_i \circ \Sigma_i \qquad \text{at } \sigma^2 = 0 \tag{2.6c}$$

$$X^\mu_{\mathrm{cl}} = X^\mu_f \circ \Sigma_f \qquad \text{at } \sigma^2 = 1. \tag{2.6d}$$

To explicitly compute the integral (2.2) we need to know the explicit form of the Teichmuller metric. Following Cohen *et al* [12] we take the following $\hat{g}_{ab}(\lambda)$:

$$\left.\begin{aligned} &\hat{g}_{11}(\lambda)(\sigma^1, \sigma^2) = 1 &(2.7a)\\ &\hat{g}_{22}(\lambda)(\sigma^1, \sigma^2) = \lambda^2 &(2.7b)\\ &\hat{g}_{12}(\lambda)(\sigma^1, \sigma^2) = 0 &(2.7c)\end{aligned}\right\} \qquad 0 < \lambda < \infty.$$

Next, we have to specify the boundary conditions defining the eigenvalue problems for the operators Δ and $P^\dagger P$. The usual type of reasoning [9, 12] leads to the following boundary conditions on contravarient worldsheet vector fields η and traceless symmetric worldsheet 2-tensors ζ_{ab}:

$$\eta^1(0, \sigma^2) = \eta^1(1, \sigma^2) = 0 \tag{2.8a}$$

$$\eta^2(\sigma^1, 0) = \eta^2(\sigma^1, 1) = 0 \tag{2.8b}$$

$$\zeta_{12}(0, \sigma^2) = \zeta_{12}(1, \sigma^2) = 0 \tag{2.9a}$$

$$\zeta_{12}(\sigma^1, 0) = \zeta_{12}(\sigma^1, 1) = 0 \tag{2.9b}$$

$$\partial_1\eta^2(0, \sigma^2) = \partial_1\eta^2(1, \sigma^2) = 0 \tag{2.10a}$$

$$\partial_2\eta^1(\sigma^1, 0) = \partial_2\eta^1(\sigma^1, 1) = 0 \tag{2.10b}$$

$$\partial_1\zeta_{11}(0, \sigma^2) = \partial_1\zeta_{11}(1, \sigma^2) = 0 \tag{2.11a}$$

$$\partial_2\zeta_{22}(\sigma^1, 0) = \partial_2\zeta_{22}(\sigma^1, 1) = 0. \tag{2.11b}$$

In equation (2.2) the dependence of the amplitude $A_E(X_i \to X_f)$ on X_i^μ and X_f^μ is obtained by splitting an arbitrary embedding X^μ into the sum $X_{cl}^\mu + \hat{X}^\mu$, where X_{cl}^μ is defined by equations (2.6a–d) and $\hat{X}^\mu$ is a shift. The functional integration over $\hat{X}^\mu$ yields the determinant of the scalar Laplacian. As X^μ and X_{cl}^μ obey the same boundary conditions, $\hat{X}^\mu$ must obey the mixed Dirichlet–Neumann boundary conditions

$$\hat{X}^\mu(\sigma^1, 0) = \hat{X}^\mu(\sigma^1, 1) = 0 \tag{2.12a}$$

$$\partial_1 \hat{X}^\mu(0, \sigma^2) = \partial_1 \hat{X}^\mu(1, \sigma^2) = 0. \tag{2.12b}$$

For a detailed discussion see [14].

The evaluation of the propagator (2.2) now proceeds parallel to the evaluation of the free closed-string propagator (sum over worldsheets with the topology of a cylinder) of [12]. We find

$$\begin{aligned} A_E(X_i \to X_f) &= \int d\Sigma_i \, d\Sigma_f \int_0^\infty d\lambda \, \lambda^{-13} \prod_{n=1}^\infty [1 - \exp(-2\pi n\lambda)]^{-12} \\ &\quad + \exp\left[\left(-\frac{1}{2\lambda}(X_{f,0} - X_{i,0})^2 - \sum_{m=1}^\infty \frac{\pi m}{4\sinh(\pi m\lambda)}\right.\right. \\ &\quad \left.\left. \times [(X_{f,m}^2 + X_{i,m}^2)\cosh(\pi m\lambda) - 2X_{f,m}X_{i,m}] + \pi\lambda\right)\right]. \end{aligned} \tag{2.13}$$

This can be rewritten as an operator expression. The Euclidean propagator for a free Newtonian particle of mass π (in 26 spatial dimensions) to go from $X_{i,0}^\mu$ to $X_{f,0}^\mu$ in time $\pi\lambda$ is [18]

$$\langle X_{f,0} | \exp\left(-\pi\lambda \frac{\hat{p}_0^2}{2\pi}\right) | X_{i,0} \rangle = (2\pi\lambda)^{-13} \exp\left(-\frac{1}{2\lambda}(X_{f,0} - X_{i,0})^2\right) \tag{2.14}$$

where $\hat{p}_0^\mu$ is the momentum conjugate to the particle's position $\hat{X}_0^\mu$. (Carets denote operators.) For a particle in a harmonic oscilator potential with angular frequency $\omega = m$:

$$\begin{aligned} \langle X_{f,m} | \exp(-\pi\lambda \hat{H}_m) | X_{i,m} \rangle &= \left(\frac{m}{1 - \exp(-2\pi m\lambda)}\right)^{13} \\ &\quad \times \exp\left(\frac{\pi m}{2\sinh(\pi m\lambda)} [(X_{f,m}^2 + X_{i,m}^2)\cosh(\pi m\lambda) - 2X_{f,m} \cdot X_{i,m}]\right) \end{aligned} \tag{2.15}$$

where

$$\hat{H}_m = \hat{p}_m^2 / 2\pi + \tfrac{1}{2}\pi m^2 \hat{X}_m^2 \tag{2.16}$$

and $\hat{p}_m^\mu$ is the momentum conjugate to the particle position $\hat{X}_m^\mu$. So (2.13) becomes

$$\begin{aligned} A_E(X_i \to X_f) &= \int d\Sigma_i \, d\Sigma_f \\ &\quad \times \int_0^\infty d\lambda \prod_{m=1}^\infty [1 - \exp(-2\pi m\lambda)] \langle X^f | \exp[-\pi\lambda(\hat{p}_0^2/2\pi + \hat{H} - 1)] | X^i \rangle \end{aligned} \tag{2.17}$$

where

$$\hat{H} = \sum_{m=1}^{\infty} \hat{H}_m \tag{2.18a}$$

$$|X^i\rangle = \prod_{m=0}^{\infty} |X_{f,m}\rangle \tag{2.18b}$$

$$\langle X^f| = \prod_{m=0}^{\infty} \langle X_{f,m}|. \tag{2.18c}$$

From [12],

$$\prod_{m=1}^{\infty} [1 - \exp(2\pi m\lambda)] = \sum_{m=-\infty}^{\infty} (-1)^m \exp[-\pi\lambda(3m^2 + m)] \tag{2.19}$$

so (2.17) can also be written as

$$\begin{aligned} A_{\mathrm{E}}(X_i \to X_f) &= \int \mathrm{d}\Sigma_i\, \mathrm{d}\Sigma_f \int_0^{\infty} \mathrm{d}\lambda \sum_{m=-\infty}^{\infty} (-1)^m \\ &\quad \times \langle X^f|\exp[-\pi\lambda(\hat{p}_0^2/2\pi + \hat{H} - 1 + 3m^2 + m)]|X^i\rangle \end{aligned} \tag{2.20}$$

or

$$A_{\mathrm{E}}(X_i \to X_f) = \int \mathrm{d}\Sigma_i\, \mathrm{d}\Sigma_f \sum_{m=-\infty}^{\infty} (-1)^m \langle X^f|(\hat{p}_0^2/2\pi + \hat{H} - 1 + 3m^2 + m)^{-1}|X^i\rangle. \tag{2.21}$$

If the initial and final states are 'pointlike', i.e.

$$X^{\mu}_{i,m} = X^{\mu}_{f,m} = 0 \qquad m \neq 0 \tag{2.22}$$

then the integration over reparametrisations of the boundary is irrelevant and may be dropped. The amplitude (2.17) becomes

$$A_{\mathrm{E}}(X_i \to X_f, \text{pointlike}) = \int_0^{\infty} \mathrm{d}\lambda \prod_{m=1}^{\infty} [1 - \exp(-2\pi m\lambda)]^{-12} \langle X^f|\exp[-\pi\lambda(\hat{p}_0^2/2\pi) - 1)]|X^i\rangle. \tag{2.23}$$

Using the Taylor expansion

$$\sum_{n=0}^{\infty} a_n z^n = \prod_{n=1}^{\infty} (1 - z^n)^{-12} \qquad a_0 = 1 \qquad a_1 = 12, \ldots, a_n \geq 0 \text{ for all } n \tag{2.24}$$

we obtain

$$A_{\mathrm{E}}(X_i \to X_f, \text{pointlike}) = \sum_{n=0}^{\infty} a_n \langle X^f|(\hat{p}_0^2/2\pi + 2n - 1)^{-1}|X^i\rangle \tag{2.25}$$

as the transition amplitude between pointlike states. This expression is identical to the one obtained if we compute the transition amplitude between pointlike states in the light cone gauge.

3. The propagator with ghosts

So far, the integrand in the path integral has been a functional of the spacetime coordinates $X^\mu(\boldsymbol{\sigma})$ and the two-dimensional metric $g_{ab}(\boldsymbol{\sigma})$. After gauge fixing, we have obtained expressions involving only the operators $\hat{p}_n^\mu$, $n \geq 0$, $\hat{X}_n^\mu$, $n > 0$, and the states $|X^i\rangle$ and $|X^f\rangle$. Ghost variables—anticommuting c-numbers and their operatorial counterparts—enter when we represent $(\det' P^\dagger P)^{1/2}$ using Grassmann integration. Although we have an explicit expression for $(\det' P^\dagger P)^{1/2}$, the representation by Grassmann integration is extremely useful, especially in string field theory [2-7].

The ghost field $c^a(\boldsymbol{\sigma})$ is a Grassmann-odd contravariant vector field. We demand that $c^a(\boldsymbol{\sigma})$ obey the boundary conditions (2.8a, b) and (2.10a, b) with $\eta^a(\boldsymbol{\sigma})$ replaced by $c^a(\boldsymbol{\sigma})$. In this way we can Fourier-expand $c^a(\boldsymbol{\sigma})$ with respect to an orthonormal basis Φ_α^a formed by the eigenfunctions of the operator $P^\dagger P$, α being an index labelling the distinct eigenmodes of $P^\dagger P$. Thus

$$c^a(\boldsymbol{\sigma}) = \Sigma_\alpha C_\alpha \Phi_\alpha{}^a(\boldsymbol{\sigma}) \tag{3.1}$$

where the C_α are Grassmann-odd numbers.

The antighost field $b_{ab}(\boldsymbol{\sigma})$ is a Grassmann-odd traceless symmetric tensor field. As with $c^a(\boldsymbol{\sigma})$, we demand that $b_{ab}(\boldsymbol{\sigma})$ obey the boundary conditions (2.9a, b) and (2.11a, b) with $\zeta_{ab}(\boldsymbol{\sigma})$ replaced by $b_{ab}(\boldsymbol{\sigma})$. This allows us to expand $b_{ab}(\boldsymbol{\sigma})$ with respect to an orthonormal basis $\Psi_{\mu ab}(\boldsymbol{\sigma})$ formed by eigenfunctions of $PP^\dagger$, μ being an index labelling the distinct eigenmodes of $PP^\dagger$. In this way we get

$$b_{ab}(\boldsymbol{\sigma}) = \Sigma_\mu B_\mu \Psi_{\mu ab}(\boldsymbol{\sigma}) \tag{3.2}$$

where the B_μ are Grassmann-odd numbers.

It can be shown [9] that $P^\dagger P$ and $PP^\dagger$ have the same non-zero eigenvalues E_α. Moreover, if Φ_α^a and $\Psi_{\alpha ab}$ correspond to the same non-zero eigenvalue E_α we may choose $\Psi_{\alpha ab}$ as follows:

$$\Psi_{\alpha ab}(\boldsymbol{\sigma}) = E_\alpha^{-1/2}(P\Phi_\alpha)_{ab}(\boldsymbol{\sigma}). \tag{3.3}$$

In this way (3.2) may be rewritten as

$$b_{ab}(\boldsymbol{\sigma}) = \Sigma_\alpha B_\alpha \Psi_{\alpha ab}(\boldsymbol{\sigma}) + \mathring{B}\Psi_{ab}(\boldsymbol{\sigma}) \tag{3.4}$$

where Ψ_{ab} is the Teichmuller deformation and the $\Psi_{\alpha ab}$ are given by (3.3). The sum is over the same set of eigenmode labels as in (3.1) because in the case we are studying $P^\dagger P$ has no zero modes.

The ghost action is given by

$$S_{\text{gh}} = -\int \mathrm{d}^2\sigma \sqrt{(\hat{g})}\hat{g}_{ab} c^a (P^\dagger b)^b. \tag{3.5a}$$

Note that in order for S_{gh} to be real under Hermitian conjugation, either b_{ab} must be Grassmann-real and c^a Grassmann-imaginary, or vice versa. We choose the convention

$$c^{a*} = -c^a \tag{3.5b}$$

$$b_{ab}^* = b_{ab}. \tag{3.5c}$$

By introducing the expansions (3.1) and (3.4) into (3.5a) we get

$$S_{\text{gh}} = -\Sigma_\alpha E_\alpha^{1/2} C_\alpha B_\alpha. \tag{3.6}$$

Note that there is no dependence on $\mathring{B}$. The ghost functional measure is

$$[\mathrm{D}c\tilde{\mathrm{D}}b] = \Pi_\alpha \, \mathrm{d}C_\alpha \, \mathrm{d}B_\alpha. \tag{3.7}$$

This measure does not contain $d\mathring{B}$. Indeed, since the ghost action (3.6) does not depend on $\mathring{B}$, the inclusion of $d\mathring{B}$ in (3.7) would make the ghost functional integral identically zero. From (3.6) and (3.7) the standard formula:

$$\int [\mathrm{D}c\tilde{\mathrm{D}}b] \exp(-S_{\mathrm{gh}}) = (\det{}' P^{\dagger}P)^{1/2} \tag{3.8a}$$

follows easily. Equivalently, we can write this as†

$$\int [\mathrm{D}c\tilde{\mathrm{D}}b]\, \mathrm{d}\mathring{B}\mathring{B} \exp(-S_{\mathrm{gh}}) \equiv \int [\mathrm{D}c\mathrm{D}b]\mathring{B} \exp(-S_{\mathrm{gh}}) = (\det{}' P^{\dagger}P)^{1/2}. \tag{3.8b}$$

To compute the propagator with ghosts we proceed as follows. We have seen in the previous section that, in order to get the propagator without ghosts, we have to integrate over all embeddings taking certain values on the boundary. Therefore it seems plausible that, in order to get the propagator with ghosts, we should integrate over all ghost and antighost fields having certain values on the boundary. We therefore implement the ghost boundary conditions by inserting Grassmann δ-functions in the Grassmann integral (3.8).

Which are the ghost boundary values mentioned above? First, since we are interested in the open-string propagator with ghosts, the boundary values should be attached to the edges $\sigma^2=0$ and $\sigma^2=1$ of the square. Moreover, the boundary conditions obeyed by $c^a(\boldsymbol{\sigma})$ and $b_{ab}(\boldsymbol{\sigma})$ constrain the value of $c^2(\boldsymbol{\sigma})$ and $b_{12}(\boldsymbol{\sigma})$ on those edges to be zero. So the only thing we can do is to assign the values of $c^1(\boldsymbol{\sigma})$ and $b_{11}(\boldsymbol{\sigma})$ on the same edges. This is compatible with the boundary conditions obeyed by $c^1(\boldsymbol{\sigma})$ and $b_{11}(\boldsymbol{\sigma})$‡. The value of $b_{22}(\boldsymbol{\sigma})$ on the boundary is determined by the tracelessness condition (in particular, it is λ dependent). Let us call $c_i \circ \Sigma_i$, $c_f \circ \Sigma_f$ the boundary values of $c^1(\boldsymbol{\sigma})$ at $\sigma^2=0, 1$, respectively. Here Σ_i and Σ_f are reparametrisations of the edges $\sigma^2=0$ and $\sigma^2=1$, respectively. Likewise, let us call $b_i \circ \Sigma_i$, $b_f \circ \Sigma_f$ the boundary values of $b_{11}(\boldsymbol{\sigma})$ at $\sigma^2=0, 1$, respectively. From (3.1)-(3.4) we get the constraints

$$0=\Gamma_i(\sigma)=\Sigma_\alpha C_\alpha \Phi^1_\alpha(\sigma^1=\sigma, 0)-c_i(\Sigma_i(\sigma)) \tag{3.9a}$$

$$0=\Gamma_f(\sigma)=\Sigma_\alpha C_\alpha \Phi^1_\alpha(\sigma^1=\sigma, 1)-c_f(\Sigma_f(\sigma)) \tag{3.9b}$$

$$0=\Delta_i(\sigma)=\Sigma_\alpha B_\alpha \psi_{\alpha 11}(\sigma^1=\sigma, 0)-b_i(\Sigma_i(\sigma)) \tag{3.10a}$$

$$0=\Delta_f(\sigma)=\Sigma_\alpha B_\alpha \psi_{\alpha 11}(\sigma^1=\sigma, 1)-b_f(\Sigma_f(\sigma)). \tag{3.10b}$$

(Note that in (3.10a, b) we have dropped the term proportional to the Teichmuller mode, as it will be set equal to zero in any case by the factor $\mathring{B}=\delta(\mathring{B})$ in (3.8b).)

Then to get the propagator with ghosts we simply have to replace $(\det{}' P^{\dagger}P)^{1/2}$ in (2.2) by the following expression:

$$A_{\mathrm{gh}} \equiv \int [\mathrm{D}c\tilde{\mathrm{D}}b] \exp(-S_{\mathrm{gh}}) \prod_{\sigma} \Gamma_i(\sigma)\Gamma_f(\sigma)\Delta_i(\sigma)\Delta_f(\sigma). \tag{3.11}$$

This is obtained by inserting the Grassmann δ functions enforcing (3.9a, b) and (3.10a, b) in the integrand on the LHS of equation (3.8). Note that there is no natural order for the δ functions. Therefore, the result is determined up to a sign. We now explicitly compute the Grassmann integral (3.11).

† Recall that $\delta(\theta)=\theta$ if θ is a Grassmann variable.

‡ See also comments preceding equation (3.39).

To begin with, we have to compute the eigenfunctions Φ^a_α and $\Psi_{\alpha ab}$. In the present case the eigenmode label α can take either the values $(m, n, 1)$ with $m > 0$ and $n \geq 0$ or the values $(m, n, 2)$ with $m \geq 0$ and $n > 0$. The eigenfunctions $\Phi_{(m,n,i)}a(\boldsymbol{\sigma})$ were computed up to normalisation in the previous section. The eigenfunctions $\Psi_{(m,n,i)ab}(\boldsymbol{\sigma})$ can be obtained by using (3.3) and (2.4a, b). With the proper normalisation we find, at $\sigma^2 = 0$ or 1,

$$\Phi^1_{(mn1)}(\sigma^1, \sigma^2) = \frac{2^{1-(\delta_{n,0})/2}}{\lambda^{1/2}}(-1)^{n\sigma^2}\sin(m\pi\sigma^1) \tag{3.12a}$$

$$\Phi^1_{(mn2)}(\sigma^1, \sigma^2) = 0 \tag{3.12b}$$

$$\Psi_{(mn1)11}(\sigma^1, \sigma^2) = \frac{2^{1-(\delta_{n,0})/2}\pi m}{\lambda^{1/2}(E_{mn}(\lambda))^{1/2}}(-1)^{n\sigma^2}\cos(m\pi\sigma^1) \tag{3.13a}$$

$$\Psi_{(mn2)11}(\sigma^1, \sigma^2) = -\frac{2^{1-(\delta_{m,0})/2}\pi n}{\lambda^{3/2}(E_{mn}(\lambda))^{1/2}}(-1)^{n\sigma^2}\cos(m\pi\sigma^1) \tag{3.13b}$$

where

$$E_{mn}(\lambda) = 2\pi^2(m^2 + n^2/\lambda^2). \tag{3.14}$$

For brevity we have listed only the relevant values entering equations (3.9a, b) and (3.10a, b). Now we consider the union F of the two edges $\sigma^2 = 0$ and $\sigma^2 = 1$. Equations (3.12a, b) and (3.13a, b) suggest that there are two convenient bases in F. They are

$$\phi_{m,\varepsilon}(\sigma^1, \sigma^2) = \varepsilon^{\sigma^2}\sin(m\pi\sigma^1) \tag{3.15}$$

$$\psi_{m,\varepsilon}(\sigma^1, \sigma^2) = \varepsilon^{\sigma^2}\cos(m\pi\sigma^1) \tag{3.16}$$

where again $\sigma^2 = 0, 1$ and $\varepsilon = \pm 1$. From (3.12–3.16), we find

$$\Phi^1_{(m,n,1)}(\sigma^1, \sigma^2) = \frac{2^{1-(\delta_{n,0})/2}\pi m}{\lambda^{1/2}}\phi_{m,(-1)}n(\sigma^1, \sigma^2) \tag{3.17}$$

$$\Psi_{(m,n,1)11}(\sigma^1, \sigma^2) = \frac{2^{1-(\delta_{n,0})/2}\pi m}{\lambda^{1/2}(E_{mn}(\lambda))^{1/2}}\psi_{m,(-1)}n(\sigma^1, \sigma^2) \tag{3.18a}$$

$$\Psi_{(m,n,2)11}(\sigma^1, \sigma^2) = -\frac{2^{1-(\delta_{m,0})/2}\pi n}{\lambda^{3/2}(E_{mn}(\lambda))^{1/2}}\psi_{m,(-1)}n(\sigma^1, \sigma^2). \tag{3.18b}$$

Denote by $c(\sigma^1, \sigma^2)$ a Grassmann-valued function *on* F such that $c(\sigma^1, 0) = c_i(\Sigma_i(\sigma^1))$ and $c(\sigma^1, 1) = c_f(\Sigma_f(\sigma^1))$. Likewise denote by $b(\sigma^1, \sigma^2)$ the Grassmann-valued function *on* F such that $b(\sigma^1, 0) = b_i(\Sigma_i(\sigma^1))$ and $b(\sigma^1, 1) = b_f(\Sigma_f(\sigma^1))$. Clearly $b(\sigma^1, \sigma^2)$ and $c(\sigma^1, \sigma^2)$ can be Fourier-expanded with respect to the bases $\phi_{m,\varepsilon}(\sigma^1, \sigma^2)$ and $\psi_{m,\varepsilon}(\sigma^1, \sigma^2)$. Using (3.9$a$, b) and (3.10a, b), respectively, and keeping in mind the boundary conditions (2.8a, b), (2.9a, b), (2.10a, b) and (2.11a, b)

$$c(\sigma^1, \sigma^2) = \sum_{m>0, \varepsilon=\pm 1} c_{m,\varepsilon}\phi_{m,\varepsilon}(\sigma^1, \sigma^2) \qquad \sigma^2 = 0, 1 \tag{3.19}$$

$$b(\sigma^1, \sigma^2) = \sum_{m\geq 0, \varepsilon=\pm 1} b_{m,\varepsilon}\psi_{m,\varepsilon}(\sigma^1, \sigma^2) \qquad \sigma^2 = 0, 1 \tag{3.20}$$

where the $c_{m,\epsilon}$ and $b_{m,\epsilon}$ are Grassmann-odd numbers. By inserting (3.17)-(3.20) into (3.9) and (3.10) we get the following constraints:

$$0=\Gamma_m^+ \equiv \sum_{n\ \text{even}\geq 0} \sum_{i=1,2} \gamma^+_{(m,n,i)} C_{(m,n,i)} - c_{m,+1} \tag{3.21a}$$

$$0=\Gamma_m^- \equiv \sum_{n\ \text{odd}>0} \sum_{i=1,2} \gamma^-_{(m,n,i)} C_{(m,n,i)} - c_{m,-1} \tag{3.21b}$$

$$0=\Delta_m^+ \equiv \sum_{n\ \text{even}\geq 0} \sum_{i=1,2} \beta^+_{(m,n,i)} B_{(m,n,i)} - b_{m,+1} \tag{3.22a}$$

$$0=\Delta_m^- \equiv \sum_{n\ \text{odd}>0} \sum_{i=1,2} \beta^-_{(m,n,i)} B_{(m,n,i)} - b_{m,-1} \tag{3.22b}$$

where

$$\gamma^\pm_{(m,n,1)} = \frac{2^{1-(\delta_{n,0})/2}}{\lambda^{1/2}} \tag{3.23a}$$

$$\gamma^\pm_{(m,n,2)} = 0 \tag{3.23b}$$

$$\beta^\pm_{(m,n,1)} = \frac{2^{1-(\delta_{n,0})/2}\pi m}{\lambda^{1/2}(E_{mn}(\lambda))^{1/2}} \tag{3.24a}$$

$$\beta^\pm_{(m,n,2)} = -\frac{2^{1-(\delta_{m,0})/2}\pi n}{\lambda^{3/2}(E_{mn}(\lambda))^{1/2}}. \tag{3.24b}$$

The forms (3.21) and (3.22) of the constraints (3.9) and (3.10) are particularly convenient because everything is expressed in terms of the Fourier coefficients $C_{(m,n,i)}$ and $B_{(m,n,i)}$ which are the Grassmann integration variables. Therefore the product $\Pi_\sigma\,(\Gamma_i\Gamma_f\Delta_i\Delta_f)$ on the RHS of (3.11) reduces simply to the products $\Pi_{m>0}\,(\Gamma_m+\Gamma_m^-)\,\Pi_{n\geq 0}\,(\Delta_n+\Delta_n^-)$. The actual calculation of the Grassmann integral (3.11) is tedious but straightforward†. The result is

$$\begin{aligned} A_{\text{gh}} &= (\det{}' P^+P)^{1/2} b_{0,+1} b_{0,-1} \\ &\times \prod_{m>0} \left[\left(c_{m,+1} b_{m,+1} + \frac{2\lambda m}{\pi} \sum_{l=0}^{\infty} \frac{2^{-\delta_{l,0}}}{m^2\lambda^2 + (2l)^2} \right) \right. \\ &\times \left. \left(c_{m,-1} b_{m,-1} + \frac{2\lambda m}{\pi} \sum_{l=0}^{\infty} \frac{1}{m^2\lambda^2 (2l+1)^2} \right) \right]. \end{aligned} \tag{3.25}$$

The determinant of P^+P has been computed in [10, equation (4.5)].

The two numerical series are easily summed [19]: (3.26a)

$$\frac{\lambda m}{\pi} \sum_{l=0}^{\infty} \frac{2^{-\delta_{l,0}}}{(\lambda m)^2 + (2l)^2} = \tfrac{1}{4}[\coth(\lambda m) + 1/\sinh(\lambda \pi m)] \tag{3.26b}$$

$$\frac{\lambda m}{\pi} \sum_{l=0}^{\infty} \frac{1}{(\lambda m)^2 + (2l+1)^2} = \tfrac{1}{4}[\coth(\lambda \pi m) - 1/\sinh(\lambda \pi m)]. \tag{3.26c}$$

† See appendix.

Further, we have to express $b_{m,\pm 1}$ and $c_{m,\pm 1}$ in terms of the Fourier coefficient of $b_i(\sigma)$, $b_f(\sigma)$, $c_i(\sigma)$ and $c_f(\sigma)$. Now

$$c_i(\Sigma_i(\sigma)) = \sum_{m>0} c^i_m \sin(m\pi\sigma) \tag{3.27a}$$

$$c_f(\Sigma_f(\sigma)) = \sum_{m>0} c^f_m \sin(m\pi\sigma) \tag{3.27b}$$

$$b_i(\Sigma_i(\sigma)) = \sum_{m\geqslant 0} b^i_m \cos(m\pi\sigma) \tag{3.28a}$$

$$b_f(\Sigma_f(\sigma)) = \sum_{m\geqslant 0} b^f_m \cos(m\pi\sigma). \tag{3.28b}$$

Here c^i_m, c^f_m, b^i_m and b^f_m are Grassmann-odd numbers depending on Σ_i and Σ_f. It is easily seen that

$$c_{m,\pm 1} = \tfrac{1}{2}(c^i_m \pm c^f_m) \tag{3.29a}$$

$$b_{m,\pm 1} = \tfrac{1}{2}(b^i_m \pm b^f_m). \tag{3.29b}$$

By plugging (3.26)-(3.29) into (3.25) we get finally

$$\begin{aligned} A_{\mathrm{gh}} = {} & \mathrm{constant} \times \lambda^{1/2} \exp\left(\frac{-\pi\lambda}{12}\right) b^i_0 b^f_0 \prod_{m=1}^{\infty} \Bigg[[1-\exp(-2\pi\lambda m)] \\ & \times \Bigg(1 - \frac{1}{\sinh(\pi\lambda m)} (c^i_m b^f_m + c^f_m b^i_m) \\ & + \coth(\pi\lambda m)(c^i_m b^i_m + c^f_m b^f_m) + c^i_m b^i_m c^f_m b^f_m \Bigg) \Bigg] \\ = {} & \mathrm{constant} \times \lambda^{1/2} \exp\left(\frac{-\pi\lambda}{12}\right) b^i_0 b^f_0 \prod_{m=1}^{\infty} \Bigg[[1-\exp(-2\pi\lambda m)] \\ & \times \exp\left(\frac{1}{\sinh(\pi\lambda m)} (\cosh(\pi\lambda m)\, (c^i_m b^i_m + c^f_m b^f_m) - c^f_m b^i_m - c^i_m b^f_m \right) \Bigg]. \end{aligned} \tag{3.30}$$

The next and final step of our calculation will be that of expressing the RHS of (3.30) in terms of ghost operators and states.

The ghost operators are $\hat{c}_m$ and $\hat{b}_m$ where m is any integer (see [20], p 128). They satisfy the anticommutation relations:

$$[\hat{b}_m, \hat{b}_n]_+ = 0 \tag{3.31a}$$

$$[\hat{b}_m, \hat{c}_n]_+ = \delta_{m+n,0} \tag{3.31b}$$

$$[\hat{c}_m, \hat{c}_n]_+ = 0. \tag{3.31c}$$

The ghost vacuum is defined by

$$\hat{c}_m |0\rangle_{\mathrm{gh}} = 0 \qquad m > 0 \tag{3.32a}$$

$$\hat{b}_m |0\rangle_{\mathrm{gh}} = 0 \qquad m \geqslant 0. \tag{3.32b}$$

The hermiticity properties are

$$\hat{c}^\dagger_m = \hat{c}_{-m} \tag{3.33a}$$

$$\hat{b}^\dagger_m = \hat{b}_{-m}. \tag{3.33b}$$

In analogy with the bosonic case [21], define

$$|w, p\rangle = \prod_{m=1}^{\infty} \exp(-w_m p_m - 2^{1/2} w_m \hat{c}_{-m} - 2^{1/2} p_m \hat{b}_{-m} - \hat{c}_{-m}\hat{b}_{-m})|0\rangle_{\text{gh}} \tag{3.34}$$

where $w_m, p_m, m > 0$, are Grassmann-odd numbers. These states have the following properties, which follow immediately from (3.31)-(3.33):

$$\frac{1}{2^{1/2}}(\hat{b}_m + \hat{b}_{-m})|w, p\rangle = w_m|w, p\rangle \tag{3.35a}$$

$$\frac{1}{2^{1/2}}(\hat{c}_m - \hat{c}_{-m})|w, p\rangle = p_m|w, p\rangle \tag{3.35b}$$

$$\langle w', p'|w, p\rangle = \prod_{m=1}^{\infty} 2[w'^*_m p_m + p'^*_m w_m - p'^*_m w'^*_m - w_m p_m]. \tag{3.35c}$$

The divergent factor $\prod_{m=1}^{\infty}(2)$ can be absorbed in the normalisation or ζ-function regularised [23]. The * denotes Hermitian conjugation. The ghost Hamiltonian is

$$\hat{H}_{\text{gh}} = \sum_{m=1}^{\infty} m(\hat{c}_{-m}\hat{b}_m + \hat{b}_{-m}\hat{c}_m). \tag{3.36}$$

From (3.31a-c) and [22] it follows that

$$\exp(-\tau\hat{H}_{\text{gh}})\hat{c}_{-m}\exp(\tau\hat{H}_{\text{gh}}) = \exp(-\tau m)\hat{c}_{-m} \tag{3.37a}$$

$$\exp(-\tau\hat{H}_{\text{gh}})\hat{b}_{-m}\exp(\tau\hat{H}_{\text{gh}}) = \exp(-\tau m)\hat{b}_{-m}. \tag{3.37b}$$

By using (3.31)-(3.34) and (3.37) we can compute the matrix element

$$\begin{aligned}\langle w', p'|\exp(-\tau\hat{H}_{\text{gh}})|w, p\rangle &= \prod_{m=1}^{\infty}\bigg[[1-\exp(-2\tau m)]\bigg(1 + (w'^*_m p_m + p'^*_m w_m)\frac{1}{\sinh(\tau m)} \\ &\quad - (p'^*_m w'^*_m + w_m p_m)\coth(\tau m) + p'^*_m w'^*_m w_m p_m\bigg)\bigg].\end{aligned} \tag{3.38a}$$

When $w'^* = w'$ and $p'^* = -p'$ this becomes

$$\begin{aligned}\langle w', p'|\exp(-\tau\hat{H}_{\text{gh}})|w, p\rangle &= \prod_{m=1}^{\infty}\bigg[[1-\exp(-\tau m)] \\ &\quad \times\bigg(1 - (p_m w'_m + p'_m w_m)\frac{1}{\sinh(\tau m)} \\ &\quad + (p'_m w'_m + p_m w_m)\coth(\tau m) + p'_m w'_m p_m w_m\bigg)\bigg] \\ &= \prod_{m=1}^{\infty}\bigg[[1-\exp(-2\tau m)]\exp\bigg(\frac{1}{\sinh(\tau m)} \\ &\quad \times[(p_m w_m + p'_m w'_m)\cosh(\tau m) - p_m w'_m - p'_m w_m]\bigg)\bigg].\end{aligned} \tag{3.38b}$$

This looks very much like (3.30) with $w_m \to b^i_m$, $w'_m \to b^f_m$, $p_m \to c^i_m$, $p'_m \to c^f_m$. (From (3.5b, c) and (3.35a, b), we see that this identification is consistent regarding the properties of these variables under Hermitian conjugation.)

We also see from (3.35a, b) and (3.31a–c) that b_m and c_m are the (Grassmann-valued) eigenvalues of operators whose anticommutator is zero. Thus it is consistent to simultaneously specify the values of all the c_m and b_m equivalently ($c(\sigma)$ and $b(\sigma)$) at the boundary. (In manipulating expressions involving matrix elements of the $\hat{c}_m$ and $\hat{b}_m$ between the coherent states (3.34), it is helpful to keep in mind the inner product (3.35c) and the fact that a Grassmann number squares to zero.)

Putting everything together we find

$$A_{\mathrm{gh}} = \text{constant } \lambda^{1/2} \exp(-\pi\lambda/12) b_0^i b_0^f \langle b^f, c^f | \exp(-\lambda\pi \hat{H}_{\mathrm{gh}}) | b^i, c^i \rangle. \tag{3.39}$$

Now we replace $(\det' P^{+}P)^{1/2}$ in equation (2.2) by A_{gh} and we develop the calculation on the same lines as in the previous section. The result is

$$\begin{aligned} A_{\mathrm{E}}(X^i, b^i, c^i \to X^f, b^f, c^f) &= \int \mathrm{d}\Sigma_i\, \mathrm{d}\Sigma_f \int_0^\infty \mathrm{d}\lambda\, b_0^i b_0^f \langle b^f, c^f | \exp(-\lambda\pi \hat{H}_{\mathrm{gh}}) | b^i, c^i \rangle \\ &\quad \times \langle X^f | \exp(-\lambda\pi(\hat{p}_0^2/2\pi + \hat{H} - 1)) | X^i \rangle \\ &= \int \mathrm{d}\Sigma_i\, \mathrm{d}\Sigma_f b_0^i b_0^f \langle X^f, b^f, c^f | (\hat{p}_0^2/2\pi + \hat{H} + \hat{H}_{\mathrm{gh}} - 1)^{-1} | X^i, b^i, c^i \rangle \end{aligned} \tag{3.40a}$$

where

$$|X^i, b^i, c^i\rangle = |X^i\rangle |b^i, c^i\rangle \tag{3.40b}$$
$$|X^f, b^f, c^f\rangle = |X^f\rangle |b^f, c^f\rangle.$$

If we integrate the amplitude $A_{\mathrm{E}}(X^i, b^i, c^i \to X^i, b^i, c^f)$ with respect to b_m^i, c_m^i, b_m^f and c_m^f we get the amplitude $A_{\mathrm{E}}(X^i \to X^f)$. Indeed, from (3.30), (3.39) and (3.40a) we have

$$\int \mathrm{d}b_0^i\, \mathrm{d}b_0^f \prod_{m=1}^{\infty} \mathrm{d}b_m^i\, \mathrm{d}b_m^f\, \mathrm{d}c_m^i\, \mathrm{d}c_m^f\, A_{\mathrm{E}}(X^i, b^i, c^i \to X^f, b^f, c^f) = A_{\mathrm{E}}(X^i \to X^f). \tag{3.41}$$

To obtain this equation we used the fact that only factors coming from the term quartic in the ghost variables with $m > 0$ in (3.30) survive under Grassmann integration in (3.41).

4. Discussion

We now wish to examine the relation between the preceding path integral computation and string field theory. To this end, we shall summarise briefly some properties of string fields [3].

An open bosonic string field is a functional $\Phi[X^\mu(\sigma), \beta(\sigma), \gamma(\sigma)]$ of a parametrised arc $X^\mu(\sigma)$ in 26-dimensional Euclidean space and two parametrised arcs in a Grassmann vector space. To comply with the conventions of the previous sections we assume that σ varies in the interval $[0, 1]$ and $X^\mu(\sigma)$ represents an open-string configuration. Thus it must satisfy the boundary condition

$$\frac{\mathrm{d}}{\mathrm{d}\sigma} X^\mu(0) = \frac{\mathrm{d}}{\mathrm{d}\sigma} X^\mu(1) = 0. \tag{4.1}$$

$\beta(\sigma)$ and $\gamma(\sigma)$ represent the configurations of the ghost field components $b_{11}(\sigma^1, \sigma^2)|_{\sigma^1=\sigma,\sigma^2=0}$ and $c^1(\sigma^1, \sigma^2)|_{\sigma^1=\sigma,\sigma^2=0}$, respectively. Thus, they satisfy the boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\beta(0) = \frac{\mathrm{d}}{\mathrm{d}\sigma}\beta(1) = 0 \tag{4.2a}$$

$$\gamma(0) = \gamma(1) = 0. \tag{4.2b}$$

From (4.2*a*) it follows that $\beta(\sigma)$ has, in general, a non-vanishing zero mode β_0, Define

$$\tilde{\beta}(\sigma) = \beta(\sigma) - \beta_0. \tag{4.3}$$

Clearly $\tilde{\beta}(\sigma)$ satisfies (4.2*a*) with $\beta(\sigma)$ replaced by $\tilde{\beta}(\sigma)$. Moreover, we can write a string field as follows:

$$\begin{aligned}\Phi[X^\mu(\sigma), \beta(\sigma), \gamma(\sigma)]\\ &= \Phi[X^\mu(\sigma), \tilde{\beta}(\sigma), \gamma(\sigma), \beta_0]\\ &= \psi[X^\mu(\sigma), \tilde{\beta}(\sigma), \gamma(\sigma)] + \beta_0\phi[X^\mu(\sigma), \tilde{\beta}(\sigma), \gamma(\sigma)].\end{aligned} \tag{4.4}$$

All physical fields are contained in ϕ [3]. In the representation we are using, the first quantised ghost operators are given by

$$\hat{b}_0 \to \beta_0 \tag{4.5a}$$

$$\hat{c}_0 \to \frac{\partial}{\partial\beta_0} \tag{4.5b}$$

The Siegel gauge constraint is

$$\hat{b}_0\Phi = \beta_0\Phi = 0. \tag{4.6}$$

From (4.4), (4.5*a*) and (4.6) we see easily that the Siegel gauge is equivalent to demanding

$$\psi = 0. \tag{4.7}$$

The two-point function of the string field Φ is, by definition,

$$G[X^f, \beta^f, \gamma^f | X^i, \beta^i, \gamma^i] = \langle\!\langle \Phi[X^f, \beta^f, \gamma^f]\Phi[X^i, \beta^i, \gamma^i]\rangle\!\rangle \tag{4.8}$$

where $\langle\!\langle\ \rangle\!\rangle$ denotes the second-quantised vacuum expectation value. In the Siegel gauge (4.6), (4.8) reduces to

$$\begin{aligned}G[X^f, \beta^f, \gamma^f | X^i, \beta^i, \gamma^i]\\ &= \beta_0^f\beta_0^i\langle\!\langle \phi[X^f, \tilde{\beta}^f, \gamma^f]\phi[X^i, \tilde{\beta}^i, \gamma^i]\rangle\!\rangle\\ &\equiv \beta_0^f\beta_0^i\Gamma[\tilde{f}; \tilde{i}].\end{aligned} \tag{4.9}$$

It is thus consistent to identify

$$\Gamma[\tilde{f}; \tilde{i}] = \langle X^f, b^f, c^f | (\hat{p}_0^2/2\pi + \hat{H} + \hat{H}_{\mathrm{gh}} - 1)^{-1} | X^i, b^i, c^i\rangle \tag{4.10}$$

(see (3.40)). The above identification can be justified as follows. $\Gamma[\tilde{f}; \tilde{i}]$ is not a reparametrisation-invariant object. Therefore, if there is a connection at all, the integration over the boundary reparametrisation in (3.40) must be dropped. Further, the labels of the states $|X^f, b^f, c^f\rangle$ and $|X^i, b^i, c^i\rangle$ match the labels $\tilde{f}$ and $\tilde{i}$, respectively. Finally, we note that the operator $(\hat{p}_0^2/2\pi) + \hat{H} + \hat{H}_{\mathrm{gh}} - 1$ is the operator $\hat{L}_0$ of the BRST Virasoro algebra. Therefore the RHS of (4.10) must contain all the propagating fields.

The above discussion is not a proof but a plausibility argument. However, it can be proved explicitly that the states (3.34) form a complete set, which implies that $\Gamma[\tilde{f};\tilde{i}]$ is indeed the propagator for the string field $\phi[X^{\mu}(\sigma),\tilde{\beta}(\sigma),\gamma(\sigma)]$. Thus the Polyakov path integral, performed between initial and final states $|X^i,b^i,c^i\rangle$, $|X^f,b^f,c^f\rangle$ of definite parametrisation, provides field-theoretic Green functions in the Siegel gauge.

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Appendix

We outline here the computation which leads to equation (3.25). In the notation of §3, the path integral (3.11) becomes

$$\begin{aligned}A_{\rm gh}&=\int[\mathrm{D}c\tilde{\mathrm{D}}b]\exp(-S_{\rm gh})\Bigg[\prod_{m>0}\Gamma_m^+\Gamma_m^-\Bigg]\Bigg[\prod_{m\geqslant0}\Delta_m^+\Delta_m^-\Bigg]\\&=\Bigg[\prod_{\substack{m>0\\n\geqslant0}}\int\mathrm{d}C_{(m,n,1)}\,\mathrm{d}B_{(m,n,1)}\Bigg]\Bigg[\prod_{\substack{m\geqslant0\\n>0}}\int\mathrm{d}C_{(m,n,2)}\,\mathrm{d}B_{(m,n,2)}\Bigg]\\&\quad\times\exp\Bigg(\sum_{\substack{m>0\\n\geqslant0}}E_{mn}^{1/2}C_{(m,n,1)}B_{(m,n,1)}+\sum_{\substack{m\geqslant0\\n>0}}E_{mn}^{1/2}C_{(m,n,2)}B_{(m,n,2)}\Bigg)\\&\quad\times\Bigg[\prod_{m>0}\Gamma_m^+\Gamma_m^-\Bigg]\Bigg[\prod_{m\geqslant0}\Delta_m^+\Delta_m^-\Bigg].\end{aligned}\tag{A1}$$

It proves convenient to write (A1) as the product of three factors:

$$A_{\rm gh}=J_0J_+J_-.\tag{A2}$$

J_0 is the product of all the factors in (A1) in which the index m is zero; J_+ is the product of all the factors in (A1) in which the index m is positive and the index n is even and J_- is the product of all the factors in (A1) in which the index m is positive and the index n is odd. That is,

$$\begin{aligned}J_0&=\Bigg(\prod_{n>0}\int\mathrm{d}C_{(0,n,2)}\,\mathrm{d}B_{(0,n,2)}\Bigg)\exp\Bigg(\sum_{n>0}E_{0n}^{1/2}C_{(0,n,2)}B_{(0,n,2)}\Bigg)\\&\quad\times\Bigg(\sum_{n\geqslant0,\mathrm{even}}\sum_{i=1,2}\beta^+_{(0,n,i)}B_{(0,n,i)}-b_{0,+}\Bigg)\\&\quad\times\Bigg(\sum_{n>0,\ \mathrm{odd}}\sum_{i=1,2}\beta^+_{(0,n,i)}B_{(0,n,i)}-b_{0,-}\Bigg)\end{aligned}\tag{A3}$$

$$J_+ = \left(\prod_{m>0} \prod_{n\geq 0,\,\text{even}} \int \mathrm{d}C_{(m,n,1)}\,\mathrm{d}B_{(m,n,1)} \right) \left(\prod_{m>0} \prod_{n>0,\,\text{even}} \int \mathrm{d}C_{(m,n,2)}\,\mathrm{d}B_{(m,n,2)} \right)$$

$$\times \exp\left(\sum_{m>0} \sum_{n\geq 0,\,\text{even}} E_{mn}^{1/2} C_{(m,n,1)} B_{(m,n,1)} + \sum_{m>0} \sum_{n>0,\,\text{even}} E_{mn}^{1/2} C_{(m,n,2)} B_{(m,n,2)} \right) \left[\prod_{m>0} \left(\sum_{n\geq 0,\,\text{even}} \gamma^{+}_{(m,n,1)} C_{(m,n,1)} - c_{m,+} \right) \right]$$

$$\times \left[\prod_{m>0} \left(\sum_{n\geq 0,\,\text{even}} \sum_{i=1,2} \beta^{+}_{(m,n,i)} B_{(m,n,i)} - b_{m,+} \right) \right] \tag{A4}$$

$$J_- = \left(\prod_{m>0} \prod_{n>0,\,\text{odd}} \int \mathrm{d}C_{(m,n,1)}\,\mathrm{d}B_{(m,n,1)} \right) \left(\prod_{m>0} \prod_{n>0,\,\text{odd}} \int \mathrm{d}C_{(m,n,2)}\,\mathrm{d}B_{(m,n,2)} \right)$$

$$\times \exp\left(\sum_{m>0} \sum_{n>0,\,\text{odd}} E_{mn}^{1/2} C_{(m,n,1)} B_{(m,n,1)} + \sum_{m>0} \sum_{n>0,\,\text{odd}} E_{mn}^{1/2} C_{(m,n,2)} B_{(m,n,2)} \right)$$

$$\times \left[\prod_{m>0} \left(\sum_{n>0,\,\text{odd}} \gamma^{-}_{(m,n,1)} C_{(m,n,1)} - c_{m,-} \right) \right]$$

$$\times \left[\prod_{m>0} \left(\sum_{n>0,\,\text{odd}} \sum_{i=1,2} \beta^{-}_{(m,n,i)} B_{(m,n,i)} - b_{m,-} \right) \right]. \tag{A5}$$

Consider first J_0. Expanding the exponential in (A3):

$$J_0 = \left(\prod_{n>0} \int \mathrm{d}C_{(0,n,2)}\,\mathrm{d}B_{(0,n,2)} \right) \left[\prod_{n>0} \left(1 + E_{0n}^{1/2} C_{(0,n,2)} B_{(0,n,2)} \right) \right]$$

$$\times \left(\sum_{n\geq 0,\,\text{even}} \sum_{i=1,2} \beta^{+}_{(0,n,i)} B_{(0,n,i)} - b_{0,+} \right) \left(\sum_{n>0,\,\text{odd}} \sum_{i=1,2} \beta^{+}_{(0,n,i)} B_{(0,n,i)} - b_{0,-} \right). \tag{A6}$$

The rules of Grassmann integration state that an integral vanishes unless the integrand has precisely one factor of each Grassmann variable with respect to which the itegration is being performed. So, any term in the integrand of (A6) which makes a non-zero contribution to the integral must be proportional to

$$\prod_{n>0} C_{(0,n,2)} B_{(0,n,2)}. \tag{A7}$$

The only such quantity is the product of all the terms involving $E_{0n}^{1/2}$ in the square bracket times the respective last terms in the last two large brackets:

$$J_0 = \left(\prod_{n>0} \int \mathrm{d}C_{(0,n,2)}\,\mathrm{d}B_{(0,n,2)} \right) \left(\prod_{n>0} E_{0n}^{1/2} C_{(0,n,2)} B_{(0,n,2)} \right) b_{0,+} b_{0,-}$$

$$= \left(\prod_{n>0} E_{0n}^{1/2} \right) b_{0,+} b_{0,-}$$

$$= b_{0,+} b_{0,-} \prod_{n>0} \left(\frac{2\pi^2 n^2}{\lambda^2} \right)^{1/2}. \tag{A8}$$

As we have in the body of this paper, we disregard in this appendix λ-independent multiplicative factors in evaluating infinite products.

Expanding the exponential in (A4), we obtain

$$J_+=\Big(\prod_{m>0}\prod_{n\geq 0,\,\text{even}}\int \mathrm{d}C_{(m,n,1)}\,\mathrm{d}B_{(m,n,1)}\Big)$$
$$\times\Big(\prod_{m>0}\prod_{n>0,\,\text{even}}\int \mathrm{d}C_{(m,n,2)}\,\mathrm{d}B_{(m,n,2)}\Big)$$
$$\times\Big(\prod_{m>0}\prod_{n\geq 0,\,\text{even}}(1+E_{mn}^{1/2}C_{(m,n,1)}B_{(m,n,1)})\Big)$$
$$\times\Big(\prod_{m>0}\prod_{n>0,\,\text{even}}(1+E_{mn}^{1/2}C_{(m,n,2)}B_{(m,n,2)})\Big)$$
$$\times\Big[\prod_{m>0}\Big(\sum_{n\geq 0,\,\text{even}}\gamma^{+}_{(m,n,1)}C_{(m,n,1)}-c_{m,+}\Big)\Big]$$
$$\times\Big[\prod_{m>0}\Big(\sum_{n\geq 0,\,\text{even}}\sum_{i=1,2}\beta^{+}_{(m,n,i)}B_{(m,n,i)}-b_{m,+}\Big)\Big]. \tag{A9}$$

For a term in the integrand of (A9) to give a non-zero contribution to the integral, it must be proportional to

$$\Big(\prod_{m>0}\prod_{n\geq 0,\,\text{even}}C_{(m,n,1)}B_{(m,n,1)}\Big)\Big(\prod_{m>0}\prod_{n>0,\,\text{even}}C_{(m,n,2)}B_{(m,n,2)}\Big). \tag{A10}$$

Terms proportional to (A10) can arise in one of two ways. Either all of the factors of $C_{(m,n,1)}$, $B_{(m,n,1)}$, $C_{(m,n,2)}$ and $B_{(m,n,2)}$ can come from the third and fourth large brackets of (A9); this gives a term proportional to

$$\Big(\prod_{m>0}\prod_{n\geq 0,\,\text{even}}E_{mn}^{1/2}\Big)\Big(\prod_{m>0}\prod_{n>0,\,\text{even}}E_{mn}^{1/2}\Big)c_{m,+}b_{m,+}. \tag{A11}$$

Or we can obtain all the factors needed for (A10) from the third and fourth large brackets in (A9), except for a single $C_{(m,n,1)}B_{(m,n,1)}$—say the one with $m=\mu$, $n=\nu$—which we get from the two square brackets. Such a term is proportional to

$$\Big(\prod_{\substack{m>0\\ m\neq\mu}}\prod_{\substack{n\geq 0,\,\text{even}\\ n\neq\nu}}E_{mn}^{1/2}\Big)\Big(\prod_{m>0}\prod_{n>0,\,\text{even}}E_{mn}^{1/2}\Big)\gamma^{+}_{(\mu,\nu,1)}\beta^{+}_{(\mu,\nu,1)}$$
$$=\Big(\prod_{m>0}\prod_{n\geq 0,\,\text{even}}E_{mn}^{1/2}\Big)\Big(\prod_{m>0}\prod_{n>0,\,\text{even}}E_{mn}^{1/2}\Big)\frac{\gamma^{+}_{(\mu,\nu,1)}\beta^{+}_{(\mu,\nu,1)}}{E_{\mu\nu}^{1/2}}. \tag{A12}$$

Thus, (A9) is equal to

$$J_+=\prod_{m>0}\Big[\Big(\prod_{n>0,\,\text{even}}E_{mn}\Big)\Big(c_{m,+}b_{m,+}+\sum_{n\geq 0,\,\text{even}}\frac{\gamma^{+}_{(m,n,1)}\beta^{+}_{(m,n,1)}}{E_{mn}^{1/2}}\Big)\Big]. \tag{A13}$$

Note that overall factors of $E_{m0}^{1/2}$ can be included or deleted as convenient, since $E_{m0}^{1/2}$ is independent of λ (see (3.14)). Using (3.14), (3.23*a*) and (3.24*a*), (A13) becomes

$$J_+=\prod_{m>0}\Big[\Big(\prod_{n>0,\,\text{even}}E_{mn}\Big)\Big(c_{m,+}b_{m,+}+\frac{\pi m}{\lambda}\sum_{n\geq 0,\,\text{even}}\frac{2^{2-\delta_{n,0}}}{2\pi^2(m^2+n^2/\lambda^2)}\Big)\Big]. \tag{A14}$$

In like manner, we obtain

$$J_-=\prod_{m>0}\Big[\Big(\prod_{n>0,\,\text{odd}}E_{mn}\Big)\Big(c_{m,-}b_{m,-}+\frac{\pi m}{\lambda}\sum_{n\geq 0,\,\text{odd}}\frac{2^{2}}{2\pi^2(m^2+n^2/\lambda^2)}\Big)\Big]. \tag{A15}$$

Using the above two equations and (A8) in (A2) yields (3.25).

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